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INEQUALITIES OF HERMITE–HADAMARD TYPE FOR FUNCTIONS OF SELFADJOINT OPERATORS AND MATRICES

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Abstract. Some inequalities of Hermite-Hadamard type for operator convex functions in Hilbert spaces are given. The case for matrices and convex functions is also considered. Examples for some particular functions of interest are provided as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}, a < b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [49]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [49]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]–[19], [22]–[29], [36]–[39] and [54].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,y \in X$, $x \neq y$. Define the segment

$$[x,y] := \{(1-t)x + ty, \, t \in [0,1]\}.$$

We consider the function $f : [x,y] \to \mathbb{R}$ and the associated function

$$g(x,y) : [0,1] \to \mathbb{R}, \; g(x,y)(t) := f[(1-t)x+ty], \; t \in [0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

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For any convex function defined on a segment $[x,y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right) \leqslant \int_0^1 f[(1-t)x+ty]dt \leqslant \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leqslant p < \infty$) is a convex function, then for any $x,y \in X$ we have the following norm inequality from (1.2) (see [55, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leqslant \int_0^1 \|(1-t)x+ty\|^p dt \leqslant \frac{\|x\|^p+\|y\|^p}{2}. \tag{1.3}$$

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [41, p. 3]):

For any $f,g \in C(\mathrm{Sp}(A))$ and any $\alpha,\beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g) = \alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(\mathrm{Sp}(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geqslant 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geqslant 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geqslant g(t) \text{ for any } t \in \mathrm{Sp}(A) \text{ implies that } f(A) \geqslant g(A) \tag{P}$$

in the operator order of $B(H)$.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A+\lambda B) \leqslant (\geqslant)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leqslant B$ with $\mathrm{Sp}(A)$, $\mathrm{Sp}(B)$ $\subset I$ imply $f(A) \leqslant f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [41] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leqslant r \leqslant 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1 \leqslant r \leqslant 2$ or $-1 \leqslant r \leqslant 0$ and is operator concave on $(0,\infty)$ if $0 \leqslant r \leqslant 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [41] and the references therein. For other results, see [52], [47], [51] and [48]. For recent results, see [23], [24], [25] and the books [27] and [28].

We recall the following result concerning a Hermite-Hadamard type inequality for operator convex functions [26] (see also [27, p. 60]):

THEOREM 1. *Let $f: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality in the operator order*

$$
\begin{aligned}
f\left(\frac{A+B}{2}\right) &\leqslant \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \\
&\leqslant \int_0^1 f((1-t)A+tB)\,dt \\
&\leqslant \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right] \leqslant \frac{f(A)+f(B)}{2}.
\end{aligned} \tag{1.4}
$$

Motivated by the above results, we establish in this paper a generalization of (1.4) as well as the corresponding trace versions for operators and matrices. Some quasilinear properties of some trace functionals associated with convex functions of matrices and applications for some instances of interest are also provided.

2. Inequalities for operator convex functions

The following representation result holds.

LEMMA 1. *Let $f: I \to \mathbb{R}$ be a continuous function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we have the equality*

$$
\begin{aligned}
\int_0^1 f[(1-t)A+tB]\,dt = (1-\lambda)\int_0^1 f[(1-t)((1-\lambda)A+\lambda B)+tB]\,dt \\
+\lambda \int_0^1 f[(1-t)A+t((1-\lambda)A+\lambda B)]\,dt.
\end{aligned} \tag{2.1}
$$

Proof. For $\lambda=0$ and $\lambda=1$ the equality (2.1) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 f[(1-t)(\lambda B+(1-\lambda)A)+tB]\,dt = \int_0^1 f[((1-t)\lambda+t)B+(1-t)(1-\lambda)A]\,dt$$

and

$$\int_0^1 f[t(\lambda B+(1-\lambda)A)+(1-t)A]\,dt = \int_0^1 f[t\lambda B+(1-\lambda t)A]\,dt.$$

If we make the change of variable $u := (1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\int_0^1 f[((1-t)\lambda+t)B+(1-t)(1-\lambda)A]\,dt = \frac{1}{1-\lambda}\int_\lambda^1 f[uB+(1-u)A]\,du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f[t\lambda B+(1-\lambda t)A]\,dt = \frac{1}{\lambda}\int_0^\lambda f[uB+(1-u)A]\,du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 f[(1-t)(\lambda B+(1-\lambda)A)+tB]\,dt \\
&\quad+\lambda\int_0^1 f[t(\lambda B+(1-\lambda)A)+(1-t)A]\,dt \\
&=\int_\lambda^1 f[uB+(1-u)A]\,du+\int_0^\lambda f[uB+(1-u)A]\,du \\
&=\int_0^1 f[uB+(1-u)A]\,du
\end{aligned}$$

and the identity (2.1) is proved. □

THEOREM 2. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned}
f\left(\frac{A+B}{2}\right) &\leqslant (1-\lambda)f\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \qquad (2.2)\\
&\leqslant \int_0^1 f[(1-t)A+tB]\,dt \\
&\leqslant \frac{1}{2}[f((1-\lambda)A+\lambda B)+(1-\lambda)f(B)+\lambda f(A)] \\
&\leqslant \frac{f(A)+f(B)}{2}.
\end{aligned}$$

Proof. Since $f : I \to \mathbb{R}$ is an operator convex function on the interval I, then by Theorem 1 we have

$$f\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right] \leqslant \int_0^1 f\left[(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right]dt \qquad (2.3)$$
$$\leqslant \frac{1}{2}\left[f\left((1-\lambda)A+\lambda B\right)+f(B)\right]$$

and

$$f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \leqslant \int_0^1 f\left[(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right]dt \qquad (2.4)$$
$$\leqslant \frac{1}{2}\left[f(A)+f\left((1-\lambda)A+\lambda B\right)\right]\int_0^1 h(t)dt.$$

Now, if we multiply the inequality (2.3) by $1-\lambda \geqslant 0$ and (2.4) by $\lambda \geqslant 0$ and add the obtained inequalities, then we get

$$(1-\lambda)f\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \qquad (2.5)$$
$$\leqslant (1-\lambda)\int_0^1 f\left[(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right]dt$$
$$+\lambda\int_0^1 f\left[(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right]dt$$
$$\leqslant \frac{1}{2}(1-\lambda)\left[f\left((1-\lambda)A+\lambda B\right)+f(B)\right]+\frac{1}{2}\lambda\left[f(A)+f\left((1-\lambda)A+\lambda B\right)\right]$$
$$=\frac{1}{2}\left[f\left((1-\lambda)A+\lambda B\right)+(1-\lambda)f(B)+\lambda f(A)\right]$$

and by (2.1) we obtain the second and the third inequalities in (2.2).

The first and the last inequality in (2.2) are obvious by operator convexity of f. $\square$

REMARK 1. If we take $\lambda = \frac{1}{2}$, then we get from (2.2) the inequality (1.4).

Some examples are as follows:

REMARK 2. Utilising different instances of operator convex or concave functions, we can provide inequalities of interest.

If $r \in [-1,0] \cup [1,2]$ then we have the inequalities for powers of operators

$$\left(\frac{A+B}{2}\right)^r \leqslant (1-\lambda)\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]^r+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^r \qquad (2.6)$$
$$\leqslant \int_0^1 \left((1-t)A+tB\right)^r dt$$
$$\leqslant \frac{1}{2}\left[\left((1-\lambda)A+\lambda B\right)^r+(1-\lambda)B^r+\lambda A^r\right]$$
$$\leqslant \frac{A^r+B^r}{2}$$

for any two selfadjoint operators A and B with spectra in $(0,\infty)$ and $\lambda \in [0,1]$.

If $r \in (0,1)$ the inequalities in (2.6) hold with "$\geqslant$" instead of "$\leqslant$".

We also have the following inequalities for logarithm

$$\begin{aligned} \ln\left(\frac{A+B}{2}\right) &\geqslant (1-\lambda)\ln\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]+\lambda\ln\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \\ &\geqslant \int_0^1 \ln((1-t)A+tB)\,dt \\ &\geqslant \frac{1}{2}\left[\ln((1-\lambda)A+\lambda B)+(1-\lambda)\ln B+\lambda\ln A\right] \\ &\geqslant \frac{\ln(A)+\ln(B)}{2} \end{aligned} \tag{2.7}$$

for any two selfadjoint operators A and B with spectra in $(0,\infty)$ and $\lambda \in [0,1]$.

Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathscr{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{2.8}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathscr{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^*f_j\|^2 \tag{2.9}$$

showing that the definition (2.8) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathscr{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathscr{B}(H)$. For $A \in \mathscr{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{2.10}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

We recall that $(\mathscr{B}_2(H),\langle\cdot,\cdot\rangle_2)$ is a complex Hilbert space, where the Hilbert-Schmidt inner product is defined by

$$\langle U,V\rangle_2 := \operatorname{tr}(V^*U),\ U,\ V \in \mathscr{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathscr{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle|A|e_i,e_i\rangle < \infty. \tag{2.11}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathscr{B}_1(H)$ the set of trace class operators in $\mathscr{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathscr{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{2.12}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.12) converges absolutely and it is independent from the choice of basis.

If $A \geqslant 0$ and $P \in \mathscr{B}_1(H)$ with $P \geqslant 0$, then

$$0 \leqslant \operatorname{tr}(PA) \leqslant \|A\| \operatorname{tr}(P). \tag{2.13}$$

Indeed, since $A \geqslant 0$, then $\langle Ax, x \rangle \geqslant 0$ for any $x \in H$. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, then

$$0 \leqslant \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \leqslant \|A\| \left\| P^{1/2}e_i \right\|^2 = \|A\| \langle Pe_i, e_i \rangle$$

for any $i \in I$. Summing over $i \in I$ we get

$$0 \leqslant \sum_{i \in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \leqslant \|A\| \sum_{i \in I} \langle Pe_i, e_i \rangle = \|A\| \operatorname{tr}(P)$$

and since

$$\sum_{i \in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle = \sum_{i \in I} \left\langle P^{1/2}AP^{1/2}e_i, e_i \right\rangle = \operatorname{tr}\left(P^{1/2}AP^{1/2}\right) = \operatorname{tr}(PA)$$

we obtain the desired result (2.13).

This obviously imply the fact that, if A and B are selfadjoint operators with $A \leqslant B$ and $P \in \mathscr{B}_1(H)$ with $P \geqslant 0$, then

$$\operatorname{tr}(PA) \leqslant \operatorname{tr}(PB). \tag{2.14}$$

Now, if A is a selfadjoint operator, then we know that

$$|\langle Ax, x \rangle| \leqslant \langle |A| x, x \rangle \text{ for any } x \in H.$$

This inequality follows by Jensen's inequality for the convex function $f(t) = |t|$ defined on a closed interval containing the spectrum of A.

If $\{e_i\}_{i \in I}$ is an orthonormal basis of H, then

$$\begin{aligned} |\operatorname{tr}(PA)| &= \left| \sum_{i \in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \right| \leqslant \sum_{i \in I} \left| \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \right| \\ &\leqslant \sum_{i \in I} \left\langle |A| P^{1/2}e_i, P^{1/2}e_i \right\rangle = \operatorname{tr}(P|A|), \end{aligned} \tag{2.15}$$

for any A a selfadjoint operator and $P \in \mathscr{B}_1(H)$ with $P \geqslant 0$.

COROLLARY 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I, for any $P \in \mathscr{B}_1(H)$ with $P \geqslant 0$, $\operatorname{tr}(P) = 1$ and for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned}
&\operatorname{tr}\left[Pf\left(\frac{A+B}{2}\right)\right] \qquad (2.16)\\
&\leqslant (1-\lambda)\operatorname{tr}\left(Pf\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]\right)+\lambda\operatorname{tr}\left(Pf\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right)\\
&\leqslant \int_0^1 \operatorname{tr}\left(Pf\left[(1-t)A+tB\right]\right)dt\\
&\leqslant \frac{1}{2}\left[\operatorname{tr}\left[Pf\left((1-\lambda)A+\lambda B\right)\right]+(1-\lambda)\operatorname{tr}\left[Pf(B)\right]+\lambda\operatorname{tr}\left[Pf(A)\right]\right]\\
&\leqslant \frac{1}{2}\left(\operatorname{tr}\left[Pf(A)\right]+\operatorname{tr}\left[Pf(B)\right]\right).
\end{aligned}$$

The proof follows by the property (2.14) and the inequality (2.2).

Similar particular inequalities of interest may be stated if one chooses various operator convex functions. However the details are not presented here.

3. Inequalities for matrices

Let M_n denote the space of $n \times n$ matrices with complex elements. Let H_n denote the $n \times n$ *Hermitian matrices*, i.e. the subset of M_n consisting of all matrices $A \in H_n$ such that $A^* = A$. There is a natural partial order on H_n : a matrix $A \in H_n$ is said to be *positive semi-definite* in case

$$\langle v, Av\rangle \geqslant 0 \text{ for all } v \in \mathbb{C}^n, \tag{3.1}$$

in which case we write $A \geqslant 0$. A is said to be *positive definite* in case the inequality (3.1) is strict for all $v \neq 0$ in $\mathbb{C}^n$, in which case we write $A > 0$. Notice that in the finite-dimensional case we have $A > 0$ if and only if $A \geqslant 0$ and A is invertible. Let H_n^+ denote the $n \times n$ positive definite matrices.

A function $f : (0,\infty) \to \mathbb{R}$ is said to be *operator monotone* in case when whenever for all n and all $A, B \in H_n^+$

$$A \geqslant B \Rightarrow f(A) \geqslant f(B).$$

The following result is well know.

THEOREM 3. *Let $f : \mathbb{R}\ (\mathbb{R}^+) \to \mathbb{R}$ be continuous and let n be a natural number. If f is monotone increasing, so is $A \mapsto \operatorname{tr}[f(A)]$ on H_n (H_n^+). Likewise, if f is convex, so is $A \mapsto \operatorname{tr}[f(A)]$ on H_n (H_n^+), and strictly so if f is strictly convex.*

The following Hermite-Hadamard type trace inequality holds.

THEOREM 4. *Let $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ be continuous convex. Then for any $A,B\in H_n$ (H_n^+) we have the inequality*

$$\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\leqslant\int_0^1\operatorname{tr}\left(f\left[(1-t)A+tB\right]\right)dt\leqslant\frac{1}{2}\left(\operatorname{tr}\left[f(A)\right]+\operatorname{tr}\left[f(B)\right]\right). \tag{3.2}$$

The proof follows by the Hermite-Hadamard type inequality (1.2) applied for the convex function $A\mapsto\operatorname{tr}\left[f(A)\right]$ defined on $H_n\ \ (H_n^+)$.

We can prove the following improvement of (3.2).

THEOREM 5. *Let $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ be continuous convex. Then for any $A,B\in H_n$ (H_n^+) and $\lambda\in[0,1]$ we have the inequality*

$$\begin{aligned}&\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right] \qquad (3.3)\\&\leqslant(1-\lambda)\operatorname{tr}\left(f\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]\right)+\lambda\operatorname{tr}\left(f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right)\\&\leqslant\int_0^1\operatorname{tr}\left(f\left[(1-t)A+tB\right]\right)dt\\&\leqslant\frac{1}{2}\left[\operatorname{tr}\left[f\left((1-\lambda)A+\lambda B\right)\right]+(1-\lambda)\operatorname{tr}\left[f(B)\right]+\lambda\operatorname{tr}\left[f(A)\right]\right]\\&\leqslant\frac{1}{2}\left(\operatorname{tr}\left[f(A)\right]+\operatorname{tr}\left[f(B)\right]\right).\end{aligned}$$

Proof. Utilising Lemma 1 we have the equality

$$\begin{aligned}&\int_0^1\operatorname{tr}\left(f\left[(1-t)A+tB\right]\right)dt \qquad (3.4)\\&=(1-\lambda)\int_0^1\operatorname{tr}\left(f\left[(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right]\right)dt\\&\quad+\lambda\int_0^1\operatorname{tr}\left(f\left[(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right]\right)dt\end{aligned}$$

for any $A,B\in H_n\ \ (H_n^+)$ and $\lambda\in[0,1]$.

Utilizing a similar argument to the one in the proof of Theorem 2 we obtain the desired result (3.3). We omit the details. $\square$

It is known that if A and B are *commuting*, i.e. $AB=BA$, then the exponential function satisfies the property

$$\exp(A)\exp(B)=\exp(B)\exp(A)=\exp(A+B).$$

Also, if X is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_a^b\exp(tX)\,dt=X^{-1}\left[\exp(bX)-\exp(aX)\right].$$

Moreover, if A and B are commuting and $B-A$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)A+sB)\,ds &= \int_0^1 \exp(s(B-A))\exp(A)\,ds\\ &= \left(\int_0^1 \exp(s(B-A))\,ds\right)\exp(A)\\ &= (B-A)^{-1}[\exp(B-A)-I_n]\exp(A)\\ &= (B-A)^{-1}[\exp(B)-\exp(A)].\end{aligned}$$

If we write the inequality (3.3) for the convex function $f(t)=\exp t$, then for any commuting $A,B\in H_n$ with $B-A$ is invertible we have

$$\begin{aligned}&\operatorname{tr}\left[\exp\left(\frac{A+B}{2}\right)\right] \quad (3.5)\\ &\leqslant (1-\lambda)\operatorname{tr}\left(\exp\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]\right)+\lambda\operatorname{tr}\left(\exp\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right)\\ &\leqslant \operatorname{tr}\left((B-A)^{-1}[\exp(B)-\exp(A)]\right)\\ &\leqslant \frac{1}{2}[\operatorname{tr}[\exp((1-\lambda)A+\lambda B)]+(1-\lambda)\operatorname{tr}[\exp(B)]+\lambda\operatorname{tr}[\exp(A)]]\\ &\leqslant \frac{1}{2}(\operatorname{tr}[\exp(A)]+\operatorname{tr}[\exp(B)]),\end{aligned}$$

for any $\lambda\in[0,1]$.

If we apply the inequality (3.3) for the convex function $f(t)=t^r$, $r\in(-\infty,0)\cup[1,\infty)$ then we have the matrix power trace inequalities

$$\begin{aligned}&\operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right] \quad (3.6)\\ &\leqslant (1-\lambda)\operatorname{tr}\left(\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]^r\right)+\lambda\operatorname{tr}\left(\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^r\right)\\ &\leqslant \int_0^1 \operatorname{tr}([(1-t)A+tB]^r)\,dt\\ &\leqslant \frac{1}{2}[\operatorname{tr}[((1-\lambda)A+\lambda B)^r]+(1-\lambda)\operatorname{tr}(B^r)+\lambda\operatorname{tr}(A^r)]\\ &\leqslant \frac{1}{2}(\operatorname{tr}(A^r)+\operatorname{tr}(B^r)),\end{aligned}$$

for any $A,B\in H_n^+$ and for any $\lambda\in[0,1]$

If $r\in(0,1)$ then the inequalities in (3.6) reverse.

If we choose in (3.6) $r=2$ and take into account that

$$\int_0^1 \operatorname{tr}\left([(1-t)A+tB]^2\right)dt=\frac{1}{3}\operatorname{tr}\left(A^2+AB+B^2\right),$$

then from (3.6) we have the quadratic inequality

$$\begin{aligned}
&\operatorname{tr}\left[\left(\frac{A+B}{2}\right)^2\right] \qquad (3.7)\\
&\leqslant (1-\lambda)\operatorname{tr}\left(\left[\frac{(1-\lambda)A+(\lambda+1)B}{2}\right]^2\right)+\lambda\operatorname{tr}\left(\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^2\right)\\
&\leqslant \frac{1}{3}\operatorname{tr}\left(A^2+AB+B^2\right)\\
&\leqslant \frac{1}{2}\left[\operatorname{tr}\left[((1-\lambda)A+\lambda B)^2\right]+(1-\lambda)\operatorname{tr}\left(B^2\right)+\lambda\operatorname{tr}\left(A^2\right)\right]\\
&\leqslant \frac{1}{2}\left(\operatorname{tr}\left(A^2\right)+\operatorname{tr}\left(B^2\right)\right),
\end{aligned}$$

for any $A,B\in H_n$ and for any $\lambda\in[0,1]$.

4. Some quasilinearity properties

Consider $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ and continuous convex function and $A,B\in H_n\ (H_n^+)$. We denote by $[A,B]$ the closed matrix segment defined by the family of matrices $\{(1-t)A+tB, t\in[0,1]\}$. We also define the trace functional

$$\Upsilon_f(A,B;t):=(1-t)\operatorname{tr}[f(A)]+t\operatorname{tr}[f(B)]-\operatorname{tr}[f((1-t)A+tB)]\geqslant 0 \qquad (4.1)$$

for any $t\in[0,1]$.

The following result concerning a trace quasilinearity property for the functional $\Upsilon_f(\cdot,\cdot;t)$ may be stated:

THEOREM 6. *Let $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ be a continuous convex function and $A,B\in H_n$ (H_n^+). Then for any $C\in\ [A,B]$ we have*

$$0\leqslant\Upsilon_f(A,C;t)+\Upsilon_f(C,B;t)\leqslant\Upsilon_f(A,B;t) \qquad (4.2)$$

for each $t\in[0,1]$, i.e., the functional $\Upsilon_f(\cdot,\cdot;t)$ is superadditive as a function of matrix interval.

If $[C,D]\subset[A,B]$, then

$$0\leqslant\Upsilon_f(C,D;t)\leqslant\Upsilon_f(A,B;t) \qquad (4.3)$$

for each $t\in[0,1]$, i.e., the functional $\Upsilon_f(\cdot,\cdot;t)$ is operator nondecreasing as a function of matrix interval.

Proof. Let $C=(1-s)A+sB$ with $s\in(0,1)$. For $t\in(0,1)$ we have

$$\begin{aligned}\Upsilon_f(C,B;t)=&(1-t)\operatorname{tr}[f((1-s)A+sB)]+t\operatorname{tr}[f(B)]\\&-\operatorname{tr}[f((1-t)[(1-s)A+sB]+tB)]\end{aligned}$$

and

$$\Upsilon_f(A,C;t) = (1-t)\operatorname{tr}[f(A)] + t\operatorname{tr}[f((1-s)A+sB)] \\ -\operatorname{tr}[f((1-t)A+t[(1-s)A+sB])]$$

giving that

$$\begin{aligned} &\Upsilon_f(A,C;t) + \Upsilon_f(C,B;t) - \Upsilon_f(A,B;t) \\ &= \operatorname{tr}[f((1-s)A+sB)] + \operatorname{tr}[f((1-t)A+tB)] \\ &\quad -\operatorname{tr}[f((1-t)(1-s)A+[(1-t)s+t]B)] - \operatorname{tr}[f((1-ts)A+tsB)]. \end{aligned} \tag{4.4}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leqslant s_1$ and $t_2 \leqslant s_2$ we have that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} \leqslant \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}. \tag{4.5}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi(t) := \frac{\varphi(t) - \varphi(a)}{t-a}$$

is monotonic nondecreasing on $I \setminus \{a\}$. Utilising this property repeatedly we have

$$\begin{aligned} \frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} &\leqslant \frac{\varphi(s_1) - \varphi(t_2)}{s_1 - t_2} = \frac{\varphi(t_2) - \varphi(s_1)}{t_2 - s_1} \\ &\leqslant \frac{\varphi(s_2) - \varphi(s_1)}{s_2 - s_1} = \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}, \end{aligned}$$

which proves the inequality (4.5).

Consider the function $\varphi : [0,1] \to \mathbb{R}$ given by $\varphi(t) := \operatorname{tr}[f((1-t)A+tB)]$. Since f is convex on I it follows that φ is convex on $[0,1]$. Now, if we consider, for given $t,s \in (0,1)$, $t_1 := ts < s =: s_1$ and $t_2 := t < t + (1-t)s =: s_2$, then $\varphi(t_1) = \operatorname{tr}[f((1-ts)A+tsB)]$ and $\varphi(t_2) = \operatorname{tr}[f((1-t)A+tB)]$ giving that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} = \operatorname{tr}\left[\frac{f((1-ts)A+tsB) - f((1-t)A+tB)}{t(s-1)}\right].$$

Also

$$\varphi(s_1) = \operatorname{tr}[f((1-s)A+sB)]$$

and

$$\varphi(s_2) = \operatorname{tr}[f((1-t)(1-s)A+[(1-t)s+t]B)]$$

giving that

$$\begin{aligned} &\frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2} \\ &= \operatorname{tr}\left[\frac{f((1-s)A+sB) - f((1-t)(1-s)A+[(1-t)s+t]B)}{t(s-1)}\right]. \end{aligned}$$

Utilising the inequality (4.5) and multiplying with $t(s-1)<0$ we deduce the following inequality

$$\begin{aligned}&\operatorname{tr}[f((1-ts)A+tsB)]-\operatorname{tr}[f((1-t)A+tB)]\\&\geqslant \operatorname{tr}[f((1-s)A+sB)]-\operatorname{tr}[f((1-t)(1-s)A+[(1-t)s+t]B)].\end{aligned}\tag{4.6}$$

Finally, by (4.4) and (4.6) we get the desired result (4.2).

Applying repeatedly the superadditivity property we have for $[C,D]\subset[A,B]$ that

$$\Upsilon_f(A,C;t)+\Upsilon_f(C,D;t)+\Upsilon_f(D,B;t)\leqslant \Upsilon_f(A,B;t)$$

giving that

$$0\leqslant \Upsilon_f(A,C;t)+\Upsilon_f(D,B;t)\leqslant \Upsilon_f(A,B;t)-\Upsilon_f(C,D;t),$$

which proves (4.3). □

For $t=\frac{1}{2}$ we consider the functional

$$\Upsilon_f(A,B):=\Upsilon_f\left(A,B;\frac{1}{2}\right)=\frac{\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]}{2}-\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\geqslant 0,$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Upsilon_f(\cdot,\cdot;t)$.

We are able then to state the following

COROLLARY 2. *Let $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ be a continuous convex function and $A,B\in H_n\ (H_n^+)$. Then we have the following bounds*

$$\begin{aligned}&\inf_{C\in[A,B]}\left[\operatorname{tr}\left[f\left(\frac{A+C}{2}\right)\right]+\operatorname{tr}\left[f\left(\frac{C+B}{2}\right)\right]-\operatorname{tr}[f(C)]\right]\\&=\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}&\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}[f(C)]+\operatorname{tr}[f(D)]}{2}-\operatorname{tr}\left[f\left(\frac{C+D}{2}\right)\right]\right]\\&=\frac{\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]}{2}-\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right].\end{aligned}\tag{4.8}$$

Proof. By the superadditivity of the functional $\Upsilon_f(\cdot,\cdot)$ we have for each $C\in[A,B]$ that

$$\begin{aligned}&\frac{\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]}{2}-\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\\&\geqslant \frac{\operatorname{tr}[f(A)]+\operatorname{tr}[f(C)]}{2}-\operatorname{tr}\left[f\left(\frac{A+C}{2}\right)\right]\\&\quad+\frac{\operatorname{tr}[f(C)]+\operatorname{tr}[f(B)]}{2}-\operatorname{tr}\left[f\left(\frac{C+B}{2}\right)\right],\end{aligned}$$

which is equivalent with

$$\operatorname{tr}\left[f\left(\frac{A+C}{2}\right)\right]+\operatorname{tr}\left[f\left(\frac{C+B}{2}\right)\right]-\operatorname{tr}\left[f(C)\right]\geqslant\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]. \tag{4.9}$$

Since the equality case in (4.9) is realized for either $C=A$ or $C=B$ we get the desired bound (4.7).

The bound (4.8) is obvious by the monotonicity of the functional $\Upsilon_f(\cdot,\cdot)$ as a function of matrix interval. □

Consider now the following functional

$$\Omega_f(A,B;t):=\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]-\operatorname{tr}[f((1-t)A+tB)]-\operatorname{tr}[f((1-t)B+tA)],$$

where, as above, $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ is a continuous convex function and $A,B\in H_n\ (H_n^+)$ while $t\in[0,1]$.

We notice that

$$\Omega_f(A,B;t)=\Omega_f(B,A;t)=\Omega_f(A,B;1-t)$$

and

$$\Omega_f(A,B;t)=\Upsilon_f(A,B;t)+\Upsilon_f(A,B;1-t)\geqslant 0$$

for any $A,B\in H_n\ (H_n^+)$ and $t\in[0,1]$.

Therefore, we can state the following result as well:

COROLLARY 3. *Let $f:\mathbb{R}\ (\mathbb{R}^+)\to\mathbb{R}$ be a continuous convex function and $A,B\in H_n\ (H_n^+)$. The functional $\Omega_f(\cdot,\cdot;t)$ is superadditive and nondecreasing as a function of matrix interval.*

In particular, if $C\in[A,B]$ then we have the inequality

$$\begin{aligned}
&\frac{1}{2}\left[\operatorname{tr}[f((1-t)A+tB)]+\operatorname{tr}[f((1-t)B+tA)]\right] \\
&\leqslant\frac{1}{2}\left[\operatorname{tr}[f((1-t)A+tC)]+\operatorname{tr}[f((1-t)C+tA)]\right] \\
&\quad+\frac{1}{2}\left[\operatorname{tr}[f((1-t)C+tB)]+\operatorname{tr}[f((1-t)B+tC)]\right]-\operatorname{tr}[f(C)].
\end{aligned} \tag{4.10}$$

Also, if $C,D\in[A,B]$ then we have the inequality

$$\begin{aligned}
&\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]-\operatorname{tr}[f((1-t)A+tB)]-\operatorname{tr}[f((1-t)B+tA)] \\
&\geqslant\operatorname{tr}[f(C)]+\operatorname{tr}[f(D)]-\operatorname{tr}[f((1-t)C+tD)]-\operatorname{tr}[f((1-t)C+tD)]
\end{aligned} \tag{4.11}$$

for any $t\in[0,1]$.

Perhaps the most interesting functional we can consider is the following one:

$$\Phi_f(A,B)=\frac{\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]}{2}-\int_0^1\operatorname{tr}[f((1-t)A+tB)]\,dt. \tag{4.12}$$

Notice that, by the second Hermite-Hadamard trace inequality for convex functions we have that $\Phi_f(A,B) \geqslant 0$.

We also observe that

$$\Phi_f(A,B) = \int_0^1 \Upsilon_f(A,B;t)\,dt = \int_0^1 \Upsilon_f(A,B;1-t)\,dt. \tag{4.13}$$

Utilising this representation, we can state the following result as well:

COROLLARY 4. *Let $f : \mathbb{R}\ (\mathbb{R}^+) \to \mathbb{R}$ be a continuous convex function and $A,B \in H_n\ (H_n^+)$. The functional $\Phi_f(\cdot,\cdot)$ is superadditive and nondecreasing as a function of matrix interval. Moreover, we have the bounds*

$$\begin{aligned} &\inf_{C\in[A,B]}\left[\int_0^1 [\operatorname{tr}[f((1-t)A+tC)] + \operatorname{tr}[f((1-t)C+tB)]]\,dt - \operatorname{tr}[f(C)]\right] \\ &= \int_0^1 \operatorname{tr}[f((1-t)A+tB)]\,dt \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}[f(C)] + \operatorname{tr}[f(D)]}{2} - \int_0^1 \operatorname{tr}[f((1-t)C+tD)]\,dt\right] \\ &= \frac{\operatorname{tr}[f(A)] + \operatorname{tr}[f(B)]}{2} - \int_0^1 \operatorname{tr}[f((1-t)A+tB)]\,dt. \end{aligned} \tag{4.15}$$

REMARK 3. The above inequalities can be applied to various concrete convex functions of interest.

If we use the inequality (4.8), then we have

$$\begin{aligned} &\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}(C^r) + \operatorname{tr}(D^r)}{2} - \operatorname{tr}\left[\left(\frac{C+D}{2}\right)^r\right]\right] \\ &= \frac{\operatorname{tr}(A^r) + \operatorname{tr}(B^r)}{2} - \operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right], \end{aligned} \tag{4.16}$$

where $r \in (-\infty,0) \cup [1,\infty)$ and $A,B \in H_n^+$.

If $r \in (0,1)$, then

$$\begin{aligned} &\sup_{C,D\in[A,B]}\left[\operatorname{tr}\left[\left(\frac{C+D}{2}\right)^r\right] - \frac{\operatorname{tr}(C^r) + \operatorname{tr}(D^r)}{2}\right] \\ &= \operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right] - \frac{\operatorname{tr}(A^r) + \operatorname{tr}(B^r)}{2}, \end{aligned} \tag{4.17}$$

for any $A,B \in H_n^+$.

We have the logarithmic bounds

$$\sup_{C,D\in[A,B]}\left[\operatorname{tr}\left[\ln\left(\frac{C+D}{2}\right)\right]-\frac{\operatorname{tr}[\ln(C)]+\operatorname{tr}[\ln(D)]}{2}\right] \tag{4.18}$$
$$=\operatorname{tr}\left[\ln\left(\frac{A+B}{2}\right)\right]-\frac{\operatorname{tr}[\ln(A)]+\operatorname{tr}[\ln(B)]}{2}$$

for any $A,B\in H_n^+$.

The following bound for the exponential also holds

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}[\exp(C)]+\operatorname{tr}[\exp(D)]}{2}-\operatorname{tr}\left[\exp\left(\frac{C+D}{2}\right)\right]\right] \tag{4.19}$$
$$=\frac{\operatorname{tr}[\exp(A)]+\operatorname{tr}[\exp(B)]}{2}-\operatorname{tr}\left[\exp\left(\frac{A+B}{2}\right)\right].$$

for any $A,B\in H_n$.

If we use the inequality (4.15), then we get the following bounds

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}(C^r)+\operatorname{tr}(D^r)}{2}-\int_0^1\operatorname{tr}[((1-t)C+tD)^r]\,dt\right] \tag{4.20}$$
$$=\frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2}-\int_0^1\operatorname{tr}[((1-t)A+tB)^r]\,dt,$$

where $r\in(-\infty,0)\cup[1,\infty)$ and $A,B\in H_n^+$.

If $r\in(0,1)$, then

$$\sup_{C,D\in[A,B]}\left[\int_0^1\operatorname{tr}[((1-t)C+tD)^r]\,dt-\frac{\operatorname{tr}(C^r)+\operatorname{tr}(D^r)}{2}\right] \tag{4.21}$$
$$=\int_0^1\operatorname{tr}[((1-t)A+tB)^r]\,dt-\frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2},$$

for any $A,B\in H_n^+$.

We also have the bound for the logarithm

$$\sup_{C,D\in[A,B]}\left[\int_0^1\operatorname{tr}[\ln((1-t)C+tD)]\,dt-\frac{\operatorname{tr}[\ln(C)]+\operatorname{tr}[\ln(D)]}{2}\right] \tag{4.22}$$
$$=\int_0^1\operatorname{tr}[\ln((1-t)A+tB)]\,dt-\frac{\operatorname{tr}[\ln(A)]+\operatorname{tr}[\ln(B)]}{2},$$

for any $A,B\in H_n^+$.

The following bound for the exponential also holds

$$\sup_{C,D\in[A,B]}\left[\frac{\operatorname{tr}[\exp(C)]+\operatorname{tr}[\exp(D)]}{2}-\int_0^1\operatorname{tr}[\exp((1-t)C+tD)]\,dt\right] \tag{4.23}$$
$$=\frac{\operatorname{tr}[\exp(A)]+\operatorname{tr}[\exp(B)]}{2}-\int_0^1\operatorname{tr}[\exp((1-t)A+tB)]\,dt,$$

for any $A,B\in H_n$.

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